

Lecture 10: Deep inelastic scattering (DIS)

High-energy electron-proton scattering (inelastic)

1. Discussion in terms of a general LT extension of the ep elastic scattering with FFs replaced by structure functions
2. DIS described by the QED interaction of a virtual photon with the constituent quarks inside the proton
3. Interpretation of experimental data:
 - quark-parton model
 - structure functions related to parton distribution functions describing the momentum distribution of the quarks

The proton turns out to be a complex system comprised of quarks, gluons and antiquarks!

The cross section for elastic ep scattering falls rapidly with energy - $\frac{d\sigma}{dQ^2} \propto \frac{1}{E_e^2}$

$\Rightarrow$ at high energy inelastic scattering dominates

$\rightarrow e^- p \rightarrow e^- X$

Invariant mass of the hadronic system:

$$W^2 = P_4^2 = (P_1 + P_2 - P_3)^2 = (q + P_2)^2$$

In elastic scattering $w^2 = p_4^2$ was always the mass of the proton m_p ! $\Rightarrow$ only ϑ angle is relevant

In inelastic scattering we have an additional degree of freedom $\Rightarrow$ two kinematic quantities are needed

Let's define the Lorentz-invariant quantities y, ν, x, Q^2 :

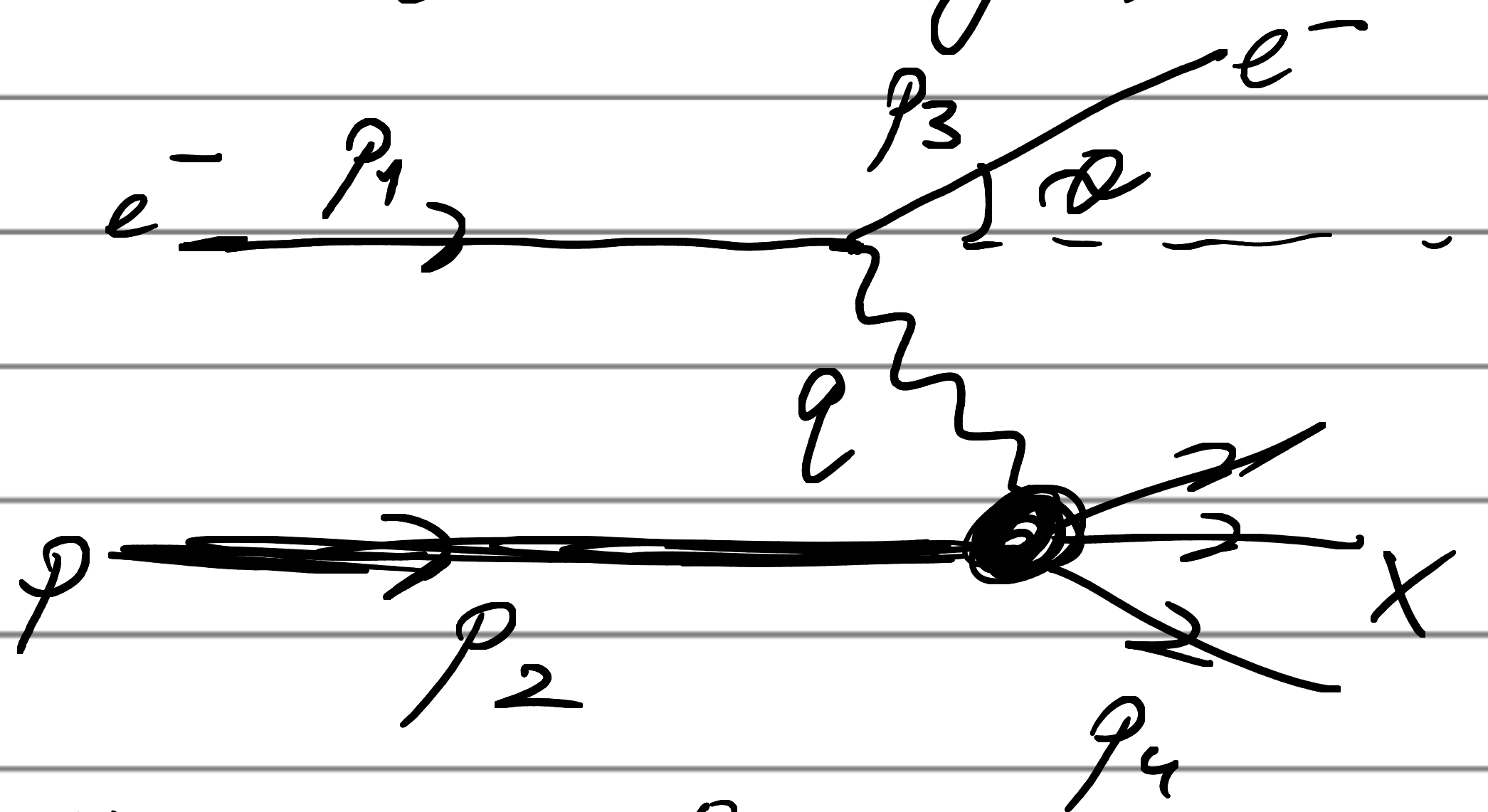
① $Q^2 = -q^2$

$$Q^2 = -(p_1 - p_3)^2 =$$

$$= -2m_e^2 + 2p_1 \cdot p_3 = -2m_e^2 + 2E_1 E_3 - 2p_1 p_3 \cos \vartheta$$

$\approx$ neglecting m_e at high-energy scattering: $Q^2 \approx 2E_1 E_3 (1 - \cos \vartheta)$

$$= 4E_1 E_3 \sin^2 \vartheta / 2 \Rightarrow \boxed{Q^2 > 0}$$



② Bjorken x :
$$x \equiv \frac{Q^2}{2p_2 \cdot q} = \frac{Q^2}{Q^2 + w^2 - m_p^2}$$

$$w^2 = p_4^2 = (q + p_2)^2 = q^2 + 2p_2 \cdot q + m_p^2 \Rightarrow 2p_2 \cdot q = w^2 + Q^2 - m_p^2$$

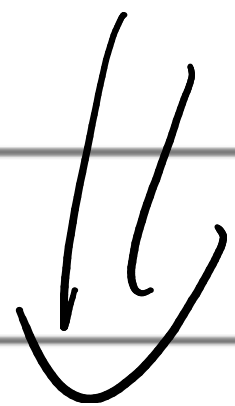
$$Q^2 > 0, w^2 \geq m_p^2 \Rightarrow \underline{0 \leq x \leq 1}$$

③ Inelasticity y :

$$y = \frac{p_2 \cdot q}{p_2 \cdot p_1}$$

$$p_2 = (m_p, 0, 0, 0) \quad p_1 = (E_1, 0, 0, E_1)$$

$$p_3 = (E_3, E_3 \sin \theta, 0, E_3 \cos \theta)$$



$$q = (E_1 - E_3, \vec{p}_1 - \vec{p}_3)$$

$$y = \frac{m_p (E_1 - E_3)}{m_p E_1} = 1 - \frac{E_3}{E_1}$$

$$0 \leq y \leq 1$$

y is the fractional energy lost by the electron in the scattering

④ Sometimes it is convenient to work with energies $\Rightarrow$ energy lost by the electron ν

$$\nu \equiv \frac{p_2 \cdot q}{m_p} = E_1 - E_3$$

Relationship between kinematic variables

$$Q^2 \equiv -q^2, \quad x \equiv \frac{Q^2}{2p_1 \cdot q}, \quad y \equiv \frac{p_2 \cdot q}{p_2 \cdot p_1}, \quad z \equiv \frac{p_2 \cdot q}{m_p}$$

$$\Rightarrow \boxed{x = \frac{Q^2}{2m_p y}}$$

$$s = (p_1 + p_2)^2 = m_p^2 + m_e^2 + 2p_1 \cdot p_2 \quad (m_p \gg m_e)$$

$$\Rightarrow 2p_1 \cdot p_2 \approx s - m_p^2$$

$$\boxed{y = \left(\frac{2m_p}{s - m_p^2} \right) z}$$

$$\boxed{Q^2 = (s - m_p^2) x \cdot y}$$

Important: for fixed CoM energy, s , the kinematics of inelastic scattering can be described by any two of the (I) quantities Q^2, z, x, y (except z and y which are not independent)

Scattering at low Q^2

-relatively low energies $\Rightarrow$ both elastic and inelastic processes

Example: Fixed-target scattering at DESY

$\theta = 10^\circ$, $E_1 = 4,879 \text{ GeV}$ from a hydrogen target

Double-Differential cross section (two variables): $\frac{d^2\sigma}{d\Omega dE_3}$

$\nearrow$ event-by-event determination

$$\Rightarrow W^2 = (p_2 + q)^2 = m_p^2 + 2p_2(p_1 - p_3) + (p_1 - p_3)^2 \approx$$

$$\approx [m_p^2 + 2m_p E_1] - 2[m_p + E_1(1 - \cos\theta)]E_3$$

(Figure 8.2 p182. Thomson)

For a fixed angle θ , linear relation between W and E_3

Complex behaviour: production of resonances, excited bound states of the proton, e.g. $\Delta^+ \rightarrow p\pi^0$

The deep-inelastic region is at large $W \rightarrow$ proton broken up and multiple-particle final states are produced

Check Fig 8.3 on p. 183 and the discussion on Q^2 -dependence of the elastic and inelastic cross sections

Deep inelastic scattering (DIS)

$e p \rightarrow e p$ elastic scattering (single p exchange) is given by the Rosenbluth formula:

$$\frac{d\sigma}{d\Omega} = \frac{\kappa^2}{4E_1^2 \sin^4 \theta/2} \cdot \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2 \theta/2 + 2\tau G_M^2 \sin^2 \theta/2 \right)$$

$$\tau = \frac{Q^2}{4m_p^2}$$

using Q^2 and y

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\kappa^2}{Q^4} \left[\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \left(1 - y - \frac{m_p^2 y^2}{Q^2} \right) + \frac{y^2}{2} G_M^2 \right]$$

absorb $G_E(Q^2)$, $G_M(Q^2)$ and τ into two new functions $f_1(Q^2)$, $f_2(Q^2)$

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\kappa^2}{Q^4} \left[\left(1 - y - \frac{m_p^2 y^2}{Q^2} \right) \underset{\substack{\uparrow \\ \text{electric + magnetic} \\ \text{interactions}}}{f_2(Q^2)} + \frac{y^2}{2} \underset{\substack{\uparrow \\ \text{magnetic interaction}}}{f_1(Q^2)} \right]$$

For elastic scattering $x=1 \Rightarrow y$ is only a function of Q^2

We can generalise to the inelastic scattering process, where the differential cross section can be expressed using x and Q^2

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\kappa^2}{Q^4} \left[\left(1 - y - \frac{m_p^2 y^2}{Q^2} \right) \frac{F_2(x, Q^2)}{x} + y^2 F_1(x, Q^2) \right]$$

General (parity-conserving) cross section. The structure functions F_1 and F_2 depend on x (can't be interpreted as Fourier transforms of the proton charge and magnetic moment distributions)

For DIS ($Q^2 \gg m_p^2 \gamma^2$) we get

$$\frac{d^2\sigma}{dx dQ^2} \approx \frac{4\pi\alpha^2}{Q^4} \left[(1-\gamma) \frac{F_2(x, Q^2)}{x} + \gamma^2 F_1(x, Q^2) \right]$$

$$Q^2 = 4E_1 E_3 \sin^2 \theta / 2, \quad x = \frac{Q^2}{2m_p(E_1 - E_3)}, \quad \gamma = 1 - \frac{E_3}{E_1}$$

The γ dependence of the cross section at different Q^2 values is used to disentangle the F_1 and F_2 contributions similar to G_E and G_M

Experimentally we can control γ by changing the energy of the incoming electron

Bjorken scaling and the Callan-Gross relation

First systematic studies of inelastic $e-p$ scattering: SLAC in California: $E_1 \in (5, 20) \text{ GeV}$

The scattering angle measured by a movable spectrometer

- $\frac{d^2\sigma}{dx dQ^2}$ measured over a range of electron energies

Several observations were made:

1. Bjorken scaling: both $F_1(x, Q^2)$ and $F_2(x, Q^2)$ are (almost) independent of Q^2 !

$$F_1(x, Q^2) \approx F_1(x) \quad \text{and} \quad F_2(x, Q^2) \approx F_2(x)$$

Suggestive of scattering from a point-like constituents within the proton!

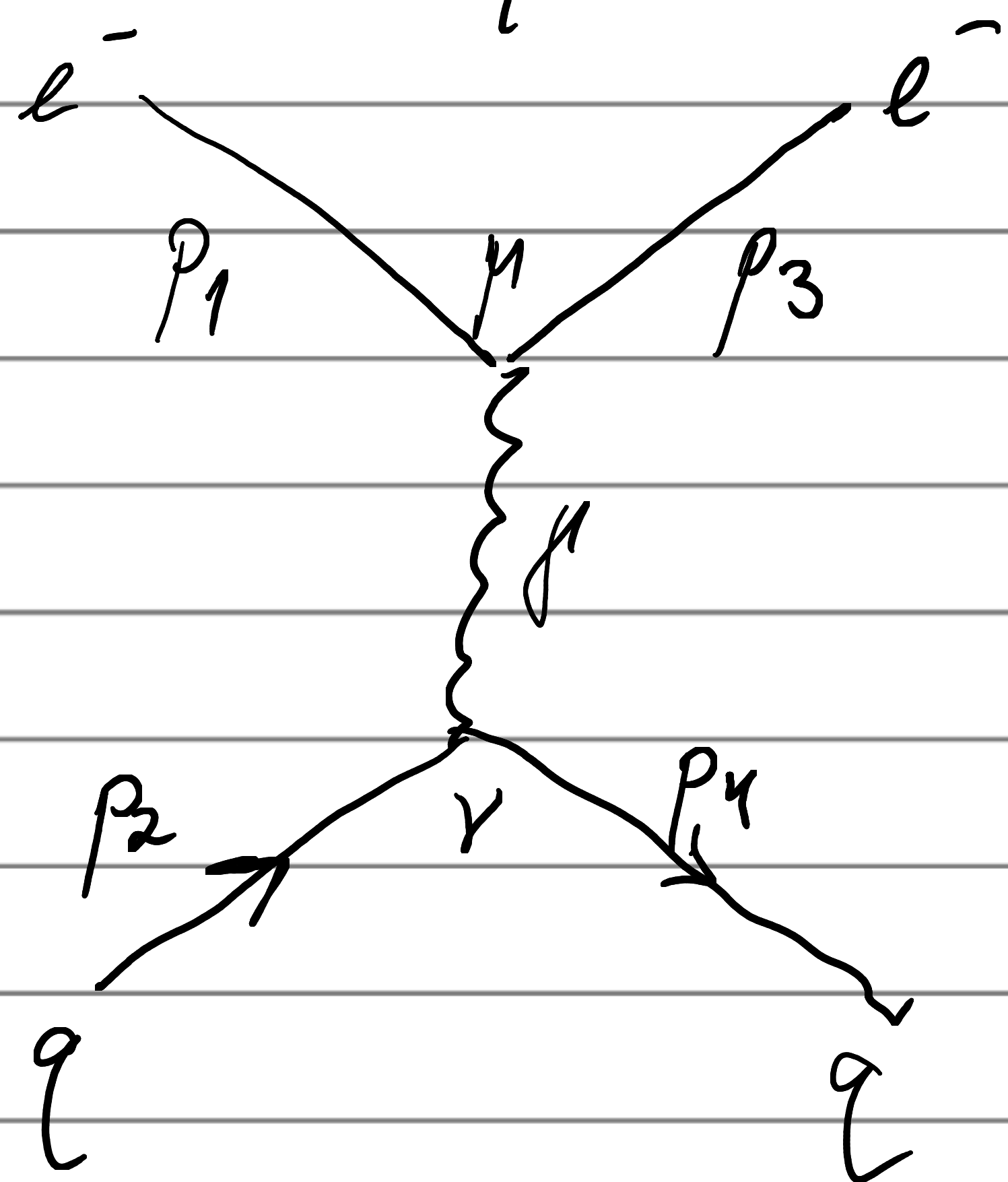
2. In DIS regime ($Q^2 > \text{few GeV}^2$) F_1 and F_2 are not independent:

$$F_2(x) = 2x F_1(x) \leftarrow \text{Callan-Gross relation}$$

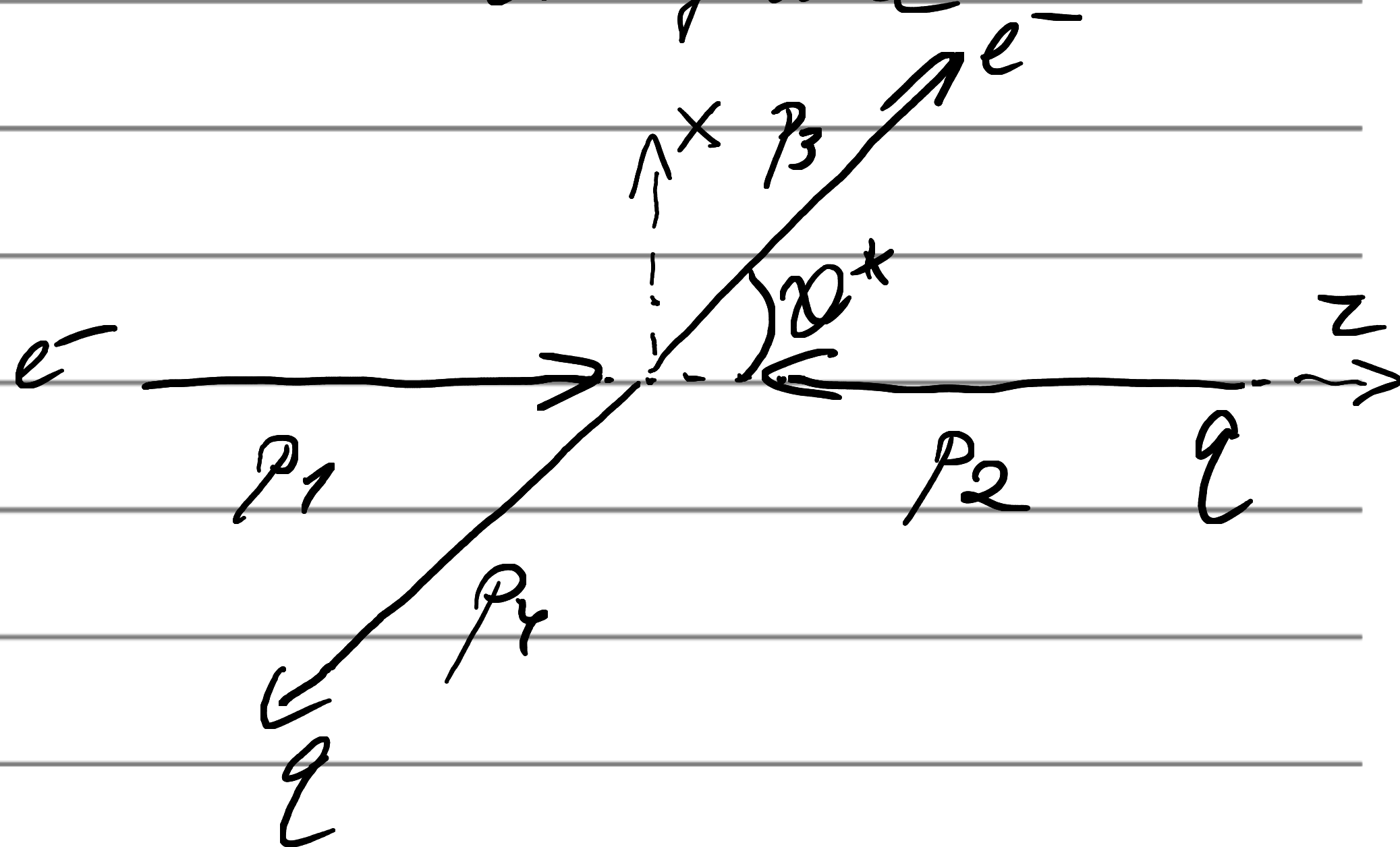
Can be explained by assuming that the underlying process in e^-p inelastic scattering is the elastic scattering from point-like spin-half constituent particles within the proton!

$\Rightarrow$ the electric and magnetic contributions relate by the fixed magnetic moment of a Dirac particle

Electron-quark scattering (QED process $e\bar{q} \rightarrow e\bar{q}$)



CoM frame



Using Feynman rules:

$$j_e^\mu = \bar{u}(p_3) [ie\gamma^\mu] u(p_1)$$

$$j_q^\nu = \bar{u}(p_4) [-iQ_q e\gamma^\nu] u(p_2)$$

Propagator $-ig_{\mu\nu}/q^2$, $q^2 = (p_1 - p_3)^2$

$$\Rightarrow M_{fi} = \frac{Q_q e^2}{q^2} [\bar{u}(p_3) \gamma^\mu u(p_1)] g_{\mu\nu} [\bar{u}(p_4) \gamma^\nu u(p_2)]$$

Use helicity amplitudes and neglect $m_e \approx m_q \approx 0$ so get the spin-averaged M_{fi} :

$$\langle |M_{fi}|^2 \rangle = 2Q_q^2 e^4 \left(\frac{s^2 + u^2}{t^2} \right) = 2Q_q^2 e^4 \frac{(p_1 \cdot p_2)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_3)^2}$$

$$\langle |M_{fi}|^2 \rangle = 2Q_g^2 e^4 \left(\frac{s^2 + u^2}{t^2} \right) = 2Q_g^2 e^4 \frac{(p_1 \cdot p_2)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_3)^2}$$

$$p_1 = (E, 0, 0, E) \quad p_2 = (E, 0, 0, -E)$$

$$p_3 = (E, +E \sin \vartheta^*, 0, +E \cos \vartheta^*), \quad p_4 = (E, -E \sin \vartheta^*, 0, -E \cos \vartheta^*)$$

$$p_1 \cdot p_2 = 2E^2 \quad p_1 \cdot p_3 = E^2(1 - \cos \vartheta^*) \quad p_1 \cdot p_4 = E^2(1 + \cos \vartheta^*)$$

$$\Rightarrow \langle |M_{fi}|^2 \rangle = 2Q_g^2 e^4 \frac{E^4 (4 + (1 + \cos \vartheta^*)^2)}{E^4 (1 - \cos \vartheta^*)^2}$$

$$\frac{d\sigma}{d\Omega^*} = \frac{Q_g^2 e^4}{8\pi^2 s} \frac{[1 + 1/4(1 + \cos \vartheta^*)^2]}{(1 - \cos \vartheta^*)^2}$$

chiral structure
of QED

arises from the $1/q^2$
propagator term

$$q^2 \rightarrow 0, \vartheta^* \rightarrow 0 \quad (\sigma \rightarrow \infty)$$

Lorentz-Invariant form: (use the expression for $\frac{d\sigma}{dt}$ ($t = q^2$))

$$\frac{d\sigma}{dq^2} = \frac{d\sigma}{d\Omega^*} \left| \frac{d\Omega^*}{dq^2} \right| = \frac{1}{64\pi s p_i^{*2}} \langle |M_{fi}|^2 \rangle = \frac{Q_g^2 e^4}{32\pi s p_i^{*2}} \left(\frac{s^2 + u^2}{t^2} \right)$$

$$p_i^* = \sqrt{s}/2 \quad \text{and} \quad t = q^2$$

$$\Rightarrow \frac{d\sigma}{dq^2} = \frac{Q_g^2 e^4}{8\pi q^4} \left(\frac{s^2 + u^2}{s^2} \right) = \frac{Q_g^2 e^4}{8\pi q^4} \left[1 + \left(\frac{u}{s} \right)^2 \right]$$

$$\frac{d\sigma}{dq^2} = \frac{Q_q^2 e^4}{8\pi q^4} \left(\frac{s^2 + u^2}{s^2} \right) = \frac{Q_q^2 e^4}{8\pi q^4} \left[1 + \left(\frac{u}{s} \right)^2 \right]$$

we can use $u \approx -s - t = -s - q^2$

$$\Rightarrow \frac{d\sigma}{dq^2} = \frac{2\pi L^2 Q_q^2}{q^4} \left[1 + \left(1 + \frac{q^2}{s} \right)^2 \right]$$

Quark-parton model (formulated for DIS when $E \gg m_p$)

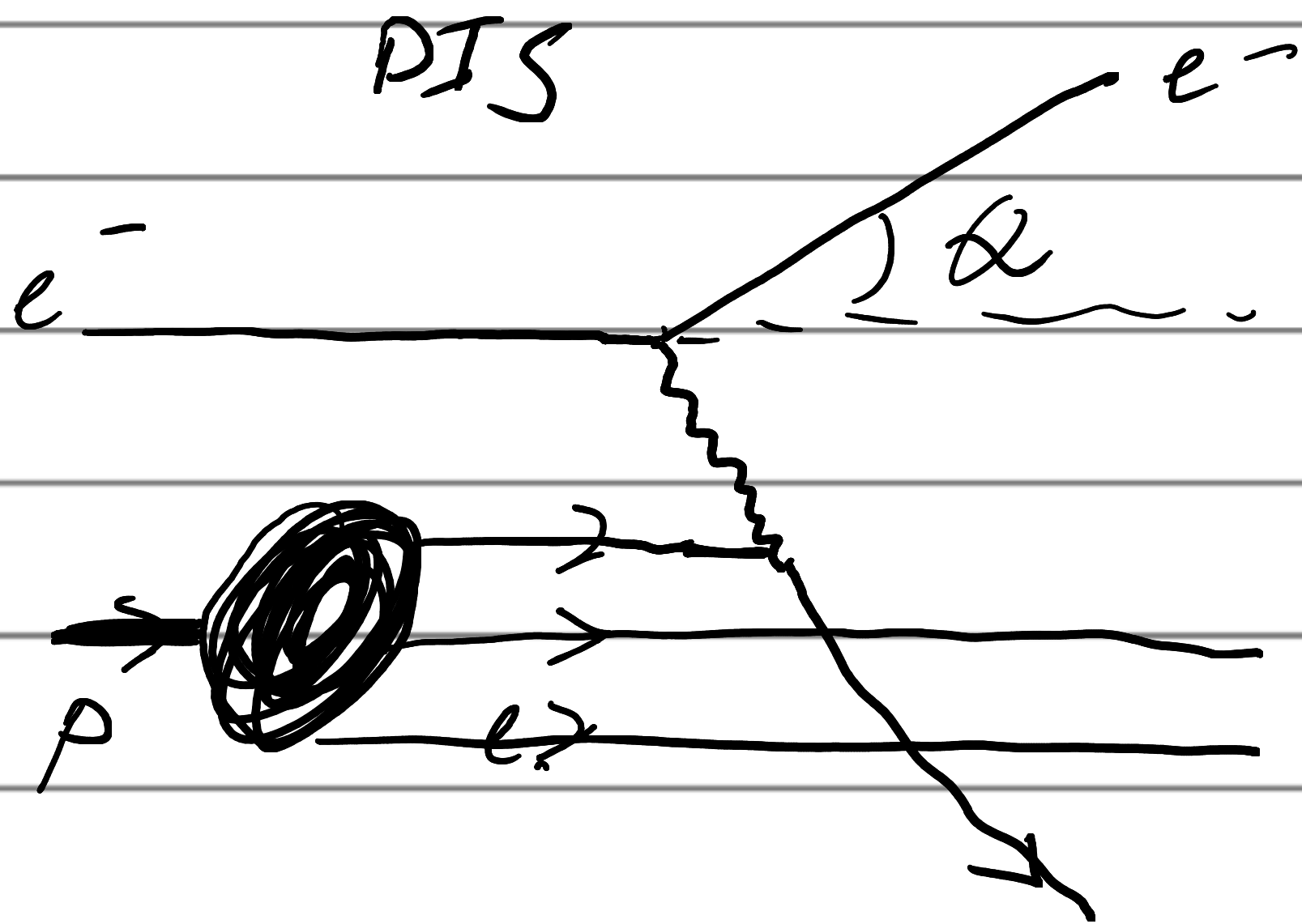
Proposed by Feynman: proton made of point-like constituents (partons)

DIS - elastic scattering from spin-half quarks inside the proton!

In this "infinite momentum (IM)" frame the proton mass can be neglected

$$p = (E_2, 0, 0, E_2)$$

Transverse momentum can also be neglected!



In the IM frame the struck quark 4-momentum is

$$p_q = \xi p = \left(\xi E_2, 0, 0, \xi E_2 \right) \quad \left\{ \begin{array}{l} \xi - \text{fraction of the proton} \\ \text{momentum carried by the} \\ \text{quark} \end{array} \right.$$

The γ -momentum of the quark after the photon interaction is:

$$\{p_2 + q \Rightarrow (\{p_2 + q)^2 = \{p_2^2 + 2\{p_2 \cdot q + q^2 = m_q^2$$

Before the interaction $\{p_2^2 = m_q^2 \Rightarrow \boxed{2\{p_2 \cdot q + q^2 = 0}$

$$\boxed{\{ = \frac{-q^2}{2p_2 \cdot q} = \frac{Q^2}{2p_2 \cdot q} = x} \quad \leftarrow \text{at very high energies}$$

In terms of the proton momentum

$$s = (p_1 + p_2)^2 \approx 2p_1 \cdot p_2 \quad y = \frac{p_2 \cdot q}{p_2 \cdot p_1} \quad x = \{ = \frac{Q^2}{2p_2 \cdot q}$$

After the interaction the struck quark's γ -momentum is $\{p_2 + q$

$$s_q = (p_1 + x p_2)^2 \approx 2x p_1 \cdot p_2 = x \cdot s \quad y_q = \frac{p_q \cdot q}{p_q \cdot p_1} = \frac{x p_2 \cdot q}{x p_2 \cdot p_1} = y$$

$$x_q = 1 \text{ (elastic scattering off the quark)}$$

We use the result for LI $e^- \mu^- \rightarrow e^- \mu^-$ scattering at high energies (ultra-relativistic limit) applied to $e^- q \rightarrow e^- q$

$$\frac{d\sigma}{dq^2} = \frac{2\pi\kappa^2 e_q^2}{q^4} \left[1 + \left(1 + \frac{q^2}{s_q} \right)^2 \right] \quad e_q = +2/3 \text{ or } -1/3$$

But we found that $Q^2 = -q^2 = (s_q - m^2) x_q y_q \Rightarrow \frac{q^2}{s_q} = -y_q = y$

$$\Rightarrow \frac{d\sigma}{dQ^2} = \frac{2\pi\kappa^2 e_q^2}{Q^4} [1 + (1 - y)^2]$$

That was a single e^-p scattering but we need to account for the distribution of parton momenta within the proton!:

- Parton Distribution Functions (PDFs): $q^p(x)dx$
 N quarks in the proton
with momentum between x and $x+dx$

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[(1-y) + \frac{y^2}{2} \right] \sum_q e_q^2 q^p(x) dx \leftarrow \text{summing over all quarks inside the proton}$$

We can measure a double differential cross section:

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[(1-y) + \frac{y^2}{2} \right] \sum_q e_q^2 q^p(x)$$

Compare with $e^-p \rightarrow e^-p$ scattering in terms of the structure functions

$$\frac{d^2\sigma^{ep}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[(1-y) \frac{F_2(Q^2, x)}{x} + \frac{y^2}{2} F_1(Q^2, x) \right]$$

$$\Rightarrow F_2^p(Q^2, x) = 2x F_1^p(Q^2, x) = x \sum_q e_q^2 q^p(x)$$

PDFs can't be calculated from QCD (α_s is large, perturbation theory doesn't work)

- we can access the PDFs by measuring the structure functions

For electron-proton DIS the structure function is related to the PDFs by:

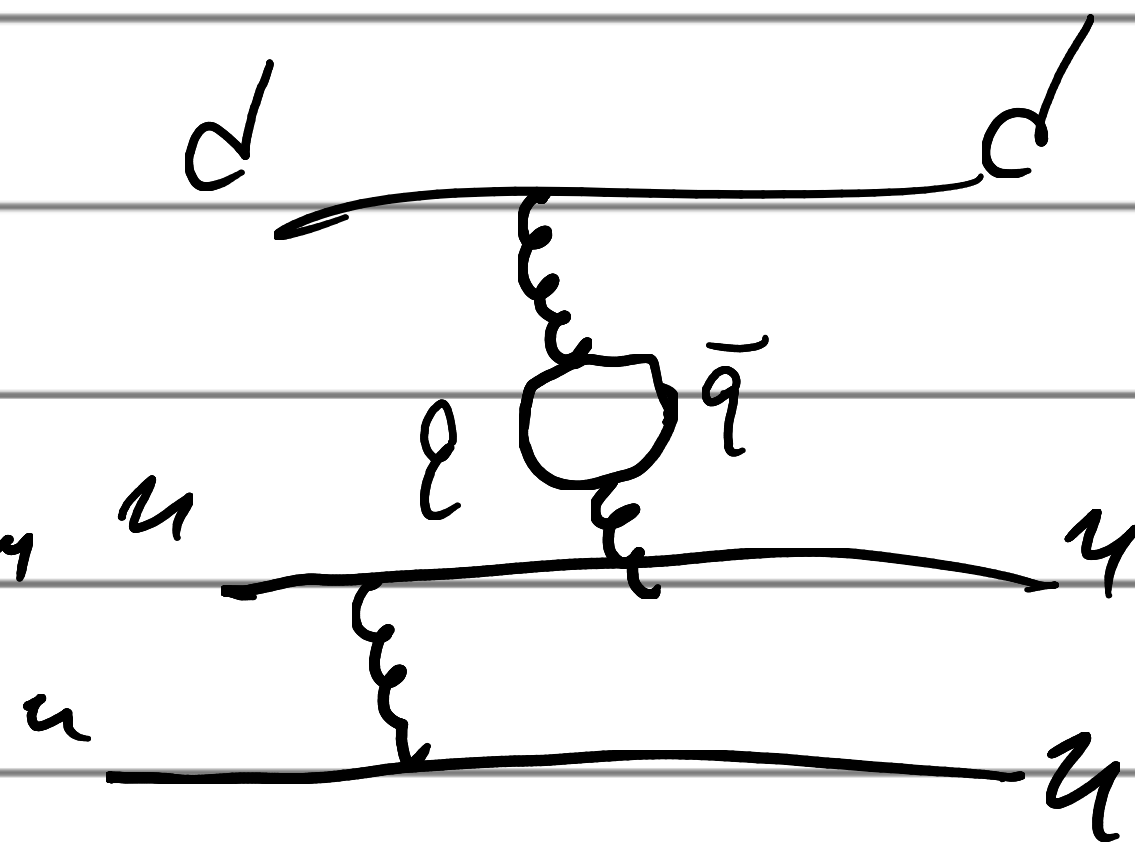
$$F_2^{ep}(x) = x \sum_i Q_i^2 q_i^p(x)$$

↑
charge

Static proton model: 3 quarks $\Rightarrow$ u u d

Dynamic proton model (much more realistic):

- constant exchange of gluons
- large momentum gluons suppressed ($1/q^2$ from each propagator)



$\Rightarrow$ sea of virtual quarks at low x

We can find all possible quarks and antiquarks inside the proton!

Slides 29+ of the lecture

Important results:

- fraction of the momentum carried by the u/u-bar is two times bigger than d/d-bar (expected by the quark-parton model)

$$f_u \approx 0,36 \quad f_d \approx 0,18$$

However, only $\approx 50\%$ of the proton momentum is carried out by the valence quarks!!

The remainder is carried by gluons but they are electrically neutral (no QED) $\Rightarrow$ we need a hadron machine!

$$Q^2 = 2m_p(E_1 - E_3) = 2m_p E_1 \gamma \quad (\text{fixed target ep scattering})$$

$$Q^2 = 4E_1 E_3 \sin^2 \theta/2 \quad \frac{\gamma^2}{Q^2} = \frac{Q^4}{Q^2} \cdot \frac{1}{4m_p^2 E_1^2} = \frac{Q^2}{4m_p^2 E_1^2} =$$

$$= \frac{E_3}{E_1} \cdot \frac{1}{m_p^2} \cdot \sin^2 \theta/2 \Rightarrow \sin^2 \theta/2 = \frac{E_1}{E_3} \cdot \frac{m_p^2 \gamma^2}{Q^2}$$

$$\sin^2 \theta/2 = 1 - \cos^2 \theta/2 \Rightarrow \cos^2 \theta/2 = 1 - \frac{E_1}{E_3} \cdot \frac{m_p^2 \gamma^2}{Q^2} \Rightarrow$$

$$\frac{E_3}{E_1} \cos^2 \theta/2 = \frac{E_3}{E_1} - \frac{m_p^2 \gamma^2}{Q^2} = 1 - \gamma - \frac{m_p^2 \gamma^2}{Q^2}$$